On Summable Positive Sequences

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Abstract

In this talk we first derive necessary conditions for a class of entire functions having only positive zeros. They are stated in terms of positive semidefinite matrices that have logarithmic derivatives of the function as elements. By applying this necessary condition to the Riemann Xi function we prove that the Riemann hypothesis is false.

Talk Outline

- Preliminaries;
- A Lemma;
- A Theorem;
- A Corollary.

Euler Γ function

- Definition:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \Re(z) > 0.$$

- Recurrence:

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \quad -z \notin \mathbb{N}_0.$$

- Singularities:

$$\mathrm{Residue}_{z=-n}\Gamma(z) = \frac{(-1)^n}{n!}, \quad n \in \mathbb{N}_0.$$

Riemann ζ,ξ functions

- Riemann Zeta Function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1.$$

- The Riemann ξ function:

$$\xi(s) = -\frac{s(1-s)}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s), \quad s \in \mathbb{C}.$$

- Functional Equation

$$\xi(s) = \xi(1-s), \quad s \in \mathbb{C}.$$

Riemann Ξ function

- Riemann Ξ function:

$$\Xi(z) = \xi\left(\frac{1}{2} + iz\right), \quad z \in \mathbb{C}.$$

- Integral representation:

$$\Xi(z) = \int_{-\infty}^{\infty} e^{-itz}\phi(t)dt = \sum_{n=0}^{\infty} \frac{(-z^2)^n}{(2n)!} \int_{-\infty}^{\infty} t^{2n}\phi(t)dt.$$

- Integrand:

$$\phi(t) \quad = \quad 2\pi \sum_{n=1}^{\infty} \left\{2\pi n^4 e^{-\frac{9t}{2}} - 3n^2 e^{-\frac{5t}{2}}\right\} \exp\left(-n^2 \pi e^{-2t}\right).$$

Some Observations

- Observation:

$$\Xi(it) > 0, \quad t \in \mathbb{R}.$$

- Observation:

$$\xi(\rho) = 0 \iff \Xi\left(\frac{\rho - \frac{1}{2}}{i}\right) = 0.$$

- Observation:

$$\rho = \frac{1}{2} + iz \iff 1 - \rho = \frac{1}{2} - iz.$$

- Conclusion: zeros z_n of $\Xi(z)$ are symmetric with respect to $\Re(z) = 0$, and $\Re(z_n) \neq 0$.

Riemann Hypothesis

Let

$$\rho_1, \rho_2, \ldots, \rho_n, \ldots$$

be the set of zeros of $\xi(s)$ with $\Im(\rho_n) > 0$, then

$$z_1, z_2, \ldots, z_n, \ldots$$

is the set of zeros with $\Re(z_n) > 0$ such that $\rho_n = \frac{1}{2} + iz_n$. The Riemann hypothesis is $\Re(\rho_n) = \frac{1}{2}, \quad n \in \mathbb{N}$ if we state it in terms of $\xi(s)$, it is $\Im(z_n) = 0, \quad n \in \mathbb{N}$ if we state it in terms of $\Xi(z)$.

Infinite Product Representation

In section 2.8 of his book, H. M. Edward essentially proved

$$\frac{\xi(s)}{\xi(\frac{1}{2})} = \prod_{\rho} \left(1 - \frac{s - \frac{1}{2}}{\rho - \frac{1}{2}}\right), \quad s \in \mathbb{C}$$

with ρ and $1 - \rho$ paired. In terms of $\Xi(z)$ is

$$\frac{\Xi(z)}{\Xi(0)} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{z_n^2}\right), \quad z \in \mathbb{C}.$$

Ingredients of the proof - a

- Series expansion:

$$\prod_{n=1}^{\infty}(1-\lambda_n z)=\sum_{n=0}^{\infty}\frac{(-z)^n}{(2n)!}\frac{\int_{-\infty}^{\infty}t^{2n}\phi(t)dt}{\int_{-\infty}^{\infty}\phi(t)dt}.$$

- Infinite product:

$$\prod_{n=1}^{\infty}(1-\lambda_n z)=\frac{\Xi(\sqrt{z})}{\Xi(0)},\quad 0\le \arg z<2\pi.$$

- Riemann hypothesis:

$$\lambda_n=\frac{1}{z_n^2}>0,\quad n\in\mathbb{N}.$$

Ingredients of the proof - b

- Riemann zeta function: For $n \in \mathbb{N}$, as $s \to +\infty$ we have

$$\zeta(s) = 1 + \mathcal{O}\left(\frac{1}{2^s}\right), \quad (\log \zeta(s))^{(n)} = \mathcal{O}\left(\frac{1}{2^s}\right).$$

- Euler's Gamma Function: as $s \to +\infty$ we have

$$\log \Gamma(z) = \frac{\log 2\pi}{2} + \left(s - \frac{1}{2}\right) \log s - s + \sum_{j=1}^{m} \frac{B_{2j}}{2j(2j-1)} \frac{1}{s^{2j-1}} + \mathcal{O}\left(\frac{1}{s^{2m+1}}\right),$$

where

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_j}{j!} z^j, \quad |z| < 2\pi.$$

Statement

Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty}|\lambda_n| < \infty$. Let

$$f(z) = \prod_{n=1}^{\infty}(1 - \lambda_n z), \quad z \in \mathbb{C}.$$

Then

$$p_k = \sum_{n=1}^{\infty}\lambda_n^k = \frac{-1}{(k-1)!}\frac{\partial^k}{\partial z^k}\log f(z)\Big|_{z=0}, \quad k \in \mathbb{N}.$$

Proof for the lemma

For $|z| \le r < \left(\sum_{n=1}^{\infty} |\lambda_n|\right)^{-1}$,

$$\frac{\partial^k}{\partial z^k} \log f(z) = -(k-1)! \sum_{n=1}^{\infty} \frac{\lambda_n^k}{(1-\lambda_n z)^k}.$$

Statement

- Assume that for $n \in \mathbb{N}$,

$$\lambda_n > 0, \quad \sum_{n=1}^{\infty} \lambda_n < \infty, \quad \sup_{n\in\mathbb{N}} \{\lambda_n\} < 1.$$

Let $f(z) = \prod_{n=1}^{\infty}(1 - \lambda_n z)$ and

$$a_{ij}(x) = \frac{-1}{(i+j+1)!} \frac{d^{i+j+1}}{dx^{i+j+1}} \frac{f'(x)}{f(x)}, \ b_{ij}(x) = \frac{-1}{(i+j)!} \frac{d^{i+j}}{dx^{i+j}} \frac{f'(x)}{f(x)},$$

for $i, j \in \mathbb{N}_0$ and $x \in \mathbb{R} \setminus \{\lambda_n^{-1}\}_{n=1}^{\infty}$. Then, for all $n \in \mathbb{N}_0$ and $x \in \mathbb{R} \setminus \{\lambda_n^{-1}\}_{n=1}^{\infty}$ the matrices $(a_{ij}(x))_{i,j=0}^{n}$ are positive semidefinite, and for all $n \in \mathbb{N}_0$ and $x < \frac{1}{\lambda_1}$ the matrices $(b_{ij}(x))_{i,j=0}^{n}$are all positive semidefinite.

Proof for the theorem-1

From

$$p_k = \frac{-1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \frac{f'(z)}{f(z)} \bigg|_{z=0}, \quad k \in \mathbb{N}$$

we get

$$\sum_{k=1}^{\infty} p_k x^k = -\sum_{k=1}^{\infty} x^k \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \frac{f'(z)}{f(z)} \bigg|_{z=0} = -x \frac{f'(x)}{f(x)}.$$

Proof for the theorem-2

Let $\mu(x)$ be

$$\mu(x) = \sum_{n=1}^{\infty} \lambda_n \delta(x - \lambda_n),$$

then for $0 < x < 1$ we have

$$p_k = \int_0^1 t^{k-1} d\mu(t), \quad \sum_{k=1}^{\infty} p_k x^k = x \int_0^1 \frac{1}{1 - xt} d\mu(t).$$

Proof for the theorem-3

Hence for $0 < x < 1$, we have

$$\int_0^1 \frac{t^n}{(1-xt)^{n+1}} d\mu(t) = -\frac{1}{n!}\frac{d^n}{dx^n}\frac{f'(x)}{f(x)}.$$

Since both sides of the last equation is analytic in $\mathbb{C} \setminus \left\{\lambda_n^{-1}\right\}_{n=1}^{\infty}$, therefore, the above equation is valid throughout $\mathbb{C} \setminus \left\{\lambda_n^{-1}\right\}_{n=1}^{\infty}$.

Proof for the theorem-4

For any $n \in \mathbb{N}_0$, $c_0, c_1, \dots, c_n \in \mathbb{R}$ and $x \in \mathbb{R} \setminus \left\{\lambda_n^{-1}\right\}_{n=1}^{\infty}$ we have

$$
\begin{aligned}
\sum_{i,j=0}^{n} a_{ij}(x) c_i c_j &= \sum_{i,j=0}^{n} \frac{-c_i c_j}{(i+j+1)!} \frac{d^{i+j+1}}{dx^{i+j+1}} \frac{f'(x)}{f(x)} \\
&= \sum_{i,j=0}^{n} c_i c_j \int_0^1 \frac{t^{i+j+1}}{(1-xt)^{i+j+2}} d\mu(t) \\
&= \int_0^1 \left(\sum_{i=0}^{n} \frac{c_i t^i}{(1-xt)^i} \right)^2 \frac{t d\mu(t)}{(1-xt)^2} \geq 0.
\end{aligned}
$$

Proof for the theorem-5

For any $n \in \mathbb{N}_0$, $c_0, c_1, \ldots, c_n \in \mathbb{R}$ and $x < \frac{1}{\lambda_1}$we have

$$\begin{aligned}
\sum_{i,j=0}^{n} b_{ij}(x) c_i c_j &= \sum_{i,j=0}^{n} \frac{-c_i c_j}{(i+j+1)!} \frac{d^{i+j}}{dx^{i+j}} \frac{f'(x)}{f(x)} \\
&= \sum_{i,j=0}^{n} c_i c_j \int_0^1 \frac{t^{i+j}}{(1-xt)^{i+j+1}} d\mu(t) \\
&= \int_0^1 \left(\sum_{i=0}^{n} \frac{c_i t^i}{(1-xt)^i} \right)^2 \frac{d\mu(t)}{(1-xt)} \geq 0.
\end{aligned}$$

Statement

The Riemann hypothesis is false.

Proof for the corollary-1

Let $x = e^{\pi i} u, \quad u > \frac{1}{4}$ we have

$$f(x) = g(u) = \frac{\Xi(i\sqrt{u})}{\Xi(0)} = \frac{\Xi(-i\sqrt{u})}{\Xi(0)} = \frac{1-4u}{\pi^{\sqrt{u}/2}} \frac{\Gamma\left(\frac{1}{4} + \frac{\sqrt{u}}{2}\right) \zeta\left(\frac{1}{2} + \sqrt{u}\right)}{\Gamma\left(\frac{1}{4}\right) \zeta\left(\frac{1}{2}\right)},$$

and

$$\begin{aligned}
a_{00}(-u) &= -\frac{d^2}{du^2} \left(\log \frac{1-4u}{\pi^{\sqrt{u}/2}} \frac{\Gamma\left(\frac{1}{4} + \frac{\sqrt{u}}{2}\right) \zeta\left(\frac{1}{2} + \sqrt{u}\right)}{\Gamma\left(\frac{1}{4}\right) \zeta\left(\frac{1}{2}\right)} \right) \\
a_{01}(-u) &= a_{10}(-u) = \frac{1}{2!} \frac{d^3}{du^3} \left(\log \frac{1-4u}{\pi^{\sqrt{u}/2}} \frac{\Gamma\left(\frac{1}{4} + \frac{\sqrt{u}}{2}\right) \zeta\left(\frac{1}{2} + \sqrt{u}\right)}{\Gamma\left(\frac{1}{4}\right) \zeta\left(\frac{1}{2}\right)} \right) \\
a_{11}(-u) &= \frac{-1}{3!} \frac{d^4}{du^4} \left(\log \frac{1-4u}{\pi^{\sqrt{u}/2}} \frac{\Gamma\left(\frac{1}{4} + \frac{\sqrt{u}}{2}\right) \zeta\left(\frac{1}{2} + \sqrt{u}\right)}{\Gamma\left(\frac{1}{4}\right) \zeta\left(\frac{1}{2}\right)} \right).
\end{aligned}$$

Proof for the corollary-2

Since as $u \to +\infty$ we have

$$\zeta\left(\frac{1}{2}+\sqrt{u}\right) = 1 + \mathcal{O}\left(\frac{1}{2\sqrt{u}}\right), \quad \log\zeta\left(\frac{1}{2}+\sqrt{u}\right) = \mathcal{O}\left(\frac{1}{2\sqrt{u}}\right),$$

then for any fixed $k \in \mathbb{N}$ and arbitrary $n > 0$

$$\frac{d^k}{du^k}\log\zeta\left(\frac{1}{2}+\sqrt{u}\right) = \mathcal{O}\left(\frac{1}{u^n}\right).$$

Proof for the corollary-3

$$
\begin{aligned}
a_{00}(-u) &= \frac{\log u - \log(4\pi^2) - 2}{16u^{3/2}} + \frac{7}{8u^2} \\
&+ \frac{1}{64u^{5/2}} + \frac{9}{16u^3} + \mathcal{O}\left(\frac{1}{u^{7/2}}\right), \\
a_{01}(-u) &= a_{10}(-u) = \frac{3\log u - \log\left(64\pi^6\right) - 8}{64u^{5/2}} + \frac{7}{8u^3} + \mathcal{O}\left(\frac{1}{u^{7/2}}\right), \\
a_{11}(-u) &= \mathcal{O}\left(\frac{1}{u^{9/2}}\right),
\end{aligned}
$$

$$
\begin{aligned}
a_{00}(-u)a_{11}(-u) - a_{01}(-u)^2 = &-\frac{\left(3\log u - \log\left(64\pi^6\right) - 8\right)^2}{4096u^5} \\
&- \frac{7\left(3\log u - \log\left(64\pi^6\right) - 8\right)}{256u^{11/2}} + \mathcal{O}\left(\frac{1}{u^6}\right).
\end{aligned}
$$

Proof for the corollary-4

- The matrix $(a_{ij}(-u))_{i,j=0}^{1}$ is not positive semidefinite as $u \to \infty$.
- Thus some numbers $\lambda_k = \frac{1}{z_k^2}$ are not positive.
- Thus some zeros z_k of Riemann Xi function $\Xi(z)$ are not real.
- Thus some zeros of Riemann xi function $\xi(s)$ with real part other than $\frac{1}{2}$.
- Thus the Riemann hypothesis is false.

Thanks

Best wishes to all!

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